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INVESTIGATION OF A SOLID BODY ROTARY MOTION AS A WHOLE*

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Behavior of rotary motions of a solid body with a fixed point is investigated throughout the whole of phase space (as a whole). The system of equations that determines the relation between the configuration space topology and the global properties of solid body motions is derived.

The global qualitative investigation of rotary motions of solid bodies can be of interest in at least two cases, viz. that of study of the evolution of rotary motions of celestial bodies, and when solving the problem of solid body stabilization. In both cases it is important to know the pattern of motions throughout the phase space (for any initial conditions) for infinitely long time intervals. Numerical methods, for example provide only a discrete number of trajectories over a limited time interval. In solving the problem of solid body stabilization (which may be considered as a problem of determining the moment that would ensure asymptotic stabilization of the considered motion) we encounter the question of size of such motion asymptotic stability, and also whether it is possible to make that motion asymptotically stable as a whole. These questions can be only answered by the global qualitative analysis of rotary motions of a solid body /1-4/

Let a solid body have a fixed point. It is convenient to define the position the coordinate system attached to the solid body relative to the inertial coordinate system by the Rodrigues-Hamilton parameters $\lambda_0, \lambda_1, \lambda_2, \lambda_3$, or, what is the same, by the single quaternion $\lambda = (\lambda_0, \lambda_1, \lambda_2, \lambda_3)$. The same angular position of the body corresponds to quaternions λ and $-\lambda$. By identifying two diametrically opposite points of the unit sphere $S^3 = \{\lambda \in R^4 : \lambda_0^2 + \lambda_1^2 + \lambda_2^2 + \lambda_3^2 = 1\}$ with λ and $-\lambda$ we obtain a real projective space P^3. The quaternion components $\lambda_0, \lambda_1, \lambda_2, \lambda_3$ are uniform coordinates of a point in P^3. The addition of solid body turns corresponds to the multiplication of quaternion, and the respective group is isomorphic to the rotation group $SO\ (3)$ which is the configuration space of the solid body.

We represent the equation of motion of the solid body with a fixed point in the attached system of coordinates

$$
\begin{gathered}
I\, d\omega/dt + \omega \times I\omega = M \\
2\lambda_0^{\cdot} = -\omega_1\lambda_1 - \omega_2\lambda_2 - \omega_3\lambda_3,\ 2\lambda_1^{\cdot} = \omega_1\lambda_0 + \omega_3\lambda_2 - \omega_2\lambda_3 \\
2\lambda_2^{\cdot} = \omega_2\lambda_0 + \omega_1\lambda_3 - \omega_3\lambda_1,\ 2\lambda_3^{\cdot} = \omega_3\lambda_0 + \omega_2\lambda_1 - \omega_1\lambda_2 \\
(\omega = (\omega_1, \omega_2, \omega_3),\ \lambda = (\lambda_0, \lambda_1, \lambda_2, \lambda_3),\ M = (M_1, M_2, M_3))
\end{gathered} \tag{1}
$$

where I is the solid body inertia tensor, ω is the angular velocity vector, λ is the unit quaternion that defines the angular position of the attached coordinate system relative to the inertia system, and M is the moment of external forces.

System (1) represents a system of differential equations on the cylinder $C = R^3 \times P^3 = \{x = (\omega, \lambda) : \omega \in R^3, \lambda \in P^3\}$.

Let us consider the motion of a solid body subjected to the action of potential, dissipative, and gyroscopic forces. In that case the moment is defined by

$$M = g + G + Q \tag{2}$$

where $g = g\,(\omega, \lambda)$ is the moment of definite dissipative forces such that ωg is negative definite relative to the angular velocity ω; $G = G\,(\omega, \lambda)$ is the moment of gyroscopic forces whose power is $\omega G = 0$, and $Q = Q\,(\lambda)$ is the moment of potential forces. If $U : P^3 \to R$ is a smooth potential function, then Q is calculated using formulas

$$
\begin{gathered}
2Q_1 = -\lambda_0 U_1 + \lambda_1 U_0 - \lambda_3 U_2 + \lambda_2 U_3 \\
2Q_2 = -\lambda_0 U_2 + \lambda_2 U_0 - \lambda_1 U_3 + \lambda_3 U_1 \\
2Q_3 = -\lambda_0 U_3 + \lambda_3 U_0 - \lambda_2 U_1 + \lambda_1 U_2
\end{gathered} \tag{3}
$$

*Prikl.Matem.Mekhan.,46,No.6,pp.940-945,1982 where U_i are partial derivatives of the potential function U with respect to uniform coordinates λ_i of the real projective space.

The system total energy dissipation total law

$$dV/dt = \omega g, \quad V = (\omega, I\omega)/2 + U(\lambda) \tag{4}$$

where V is the total mechanical energy consisting of the kinetic and potential energies, applies for system (1) – (3).

The motion of a solid body subjected to dissipative, potential, and gyroscopic forces was investigated in /5–7/ in which the problems of behavior of trajectories as a whole, and of constraints imposed by the configuration space topology on the over-all picture were not considered.

The pattern of solid body motions throughout the phase space is defined by the following theorem.

Theorem. If the potential function $U: P^3 \to R$ has nondegenerate critical points, then:

1°. There is a finite even number $n \geqslant 4$ of equilibrium positions $x_0, x_1, \ldots, x_{n-1}$ all of which are hyperbolic. Let $W(x_k)$ and $W^*(x_k)$ be the stable and unstable manifold of the equilibrium position x_k, and a_i^0 be the number of equilibrium positions for which the stable manifold dimension is i, i.e. $\dim W(x_k) = i, i = 3, 4, 5, 6$. The equalities

$$\begin{gathered} a_6^0 \geqslant 1,\ a_5^0 - a_6^0 \geqslant 0,\ a_4^0 - a_5^0 + a_6^0 \geqslant 1, \\ a_3^0 - a_4^0 + a_5^0 - a_6^0 = 0 \end{gathered} \tag{5}$$

then apply.

2°. Let $U_0 < U_1 \geqslant \ldots U_{n-1}$ be the critical value of the potential function that corresponds to equilibrium positions x_k $(k = 0, 1, \ldots, n-1)$. Consider the aggregate of sets $\Omega_l = \{x \in C : V(x) < l\}$, $\Omega_\infty = \lim \Omega_l = C$. Any motion of the solid body that begins in Ω_l, remains in Ω_l and approaches to one of the equilibrium positions $x_k \in \Omega_l$. In particular, every motion that begins in Ω_{U_1} approaches x_0 (Ω_{U_1} defines the lower bound of the asymptotic stability region for the equilibrium position x_0); every motion that begins in Ω_{U_2} approaches either x_0 or x_1, etc.

For the determination of all equilibrium positions of a solid body it is, evidently, necessary and sufficient to find all critical points of the potential function $U: P^3 \to R$. Using the Morse theory /2/ it is possible to establish the relation between the number and indices of critical points of function which is smooth in P^3, and the topological invariants of the real projective space. If the critial points of U are nondegenerate, then they are isolated and, by virtue of compactness of P^3, their number is finite.

Let c_i be the number of critical points with index $i = 0, 1, 2, 3$, and in particular c_0 and c_3 be the number maxima and minima of U, respectively. Taking into account that all nontrivial Betti numbers in modulo two of the manifold P^3 are equal unity, we obtain the Morse inequalities

$$c_0 \geqslant 1,\ c_1 - c_0 \geqslant 0,\ c_2 - c_1 + c_0 \geqslant 1,\ c_3 - c_2 + c_1 - c_0 = 0 \tag{6}$$

from which immediately follows that the number of equilibrium positions of a solid body in the field of potential, dissipative, and gyroscopic forces is always even and not less than four.

The properties of solid body motions in the equilibrium position neighborhood x is completely determined by the index i of the respective critical point. In the critical point neighborhood the potential function U is represented by a nondegenerate quadratic form relative to local coordinates with a negative inertia index i. Analysis of the linearized system in the equilibrium position neighborhood shows that it has i eigenvalues with positive real part, and $6 - i$ with negative one. Hence the equilibrium position is a multidimensional hyperbolic point for which the unsteady manifold $W^*(x)$ dimensional is equal i, while that of the stable manifold $W(x)$ is equal $6 - i$. Hence $c_i = a_{6-i}^0$, and the validity of inequalities (5) follows from inequalities (6).

The second part of the theorem follows from the known statement of the theory of stability /8/ that every motion of the system approaches the maximum invariant set which converts to zero the left-hand side of ωg of formula (4). In the case of total dissipation the set is exhausted by the equilibrium positions.

The theorem provides the lower bound of the number of critical points for any arbitrary smooth function. In applied problems the potential function can be approximated by the polynomial functions

$$U = \sum_{i, j, s, t=0}^{2k} a_{i,j,s,t} \lambda_0^i \lambda_1^j \lambda_2^s \lambda_3^t \tag{7}$$

where summation is carried out over indices that satisfy the condition $i+j+s+t=2k$.

Let us estimate the limits of variation of the polynomial function critical points number. According to the theorem proof there exists a one-to-one correspondence between the function critical points and the real solutions of the system, which is obtained by equation to zero the right-hand sides of Eqs.(3). Each equation of that homogeneous system (we shall call it system A) is an even form of order $2k$. It real solution normalized with respect to unity corresponds on P^3 to a critical point of function U. By the Bezout theorem the maximum number of isolated complex solutions of the system of homogeneous algebraic equations is equal $8k^3$. An exact estimate of the number of real solutions can be obtained using the following property /of the problem/. Solutions of the system can be separated in two groups: one corresponding to the condition of proportionality of $\operatorname{grad} U$ and λ (these can be real and complex solutions) the second consisting of extraneous complex roots satisfying the condition $\lambda_0^2+\lambda_1^2+\lambda_2^2+\lambda_3^2=0$. It is important to note that the extraneous complex roots satisfy a system of equations that is independent of system A and that a continuous variation of the form coefficients cannot transform them into real ones. Thus, when some form of order $2k$ for which system A has the maximum possible number of solutions $v+u=8k^3$, where v is the number of real solutions and u is the number of extraneous complex solutions, then v is the maximum number of real solutions of system A for an arbitrary form of order $2k$. Then, if the potential function is a quadratic form, the body has four equilibrium positions.

Among fourth order forms form $U=\lambda_0^4+\lambda_1^4+\lambda_2^4+\lambda_3^4$ has the indicated property. In that case system A has the maximum number of roots equal 64 of which 40 are real and 24 are extraneous complex. If the potential function is a fourth order form in the Rodrigues—Hamilton parameters, the number of equilibrium positions may reach forty. For the gravitation potential which is a fourth order form that number is 24. Generally, the construction of the required form is complicated.

Thus a solid body in a field of potential, dissipative, and gyroscopic forces has an asymptotically stable equilibrium position. This enables the solution of the problem of solid body stabilization /5—7/. However it is not possible to make this equilibrium position stable as a whole by using the indicated forces, since there are always at least three saddle type equilibrium positions of various types.

Let us show that, when a reasonably arbitrary moment acts on the body, there are inequalities similar to (5) generally apply.

The dynamic system specified on the cylinder $C=R^3\times P^3$, which defines the motions of a solid body, can have a number of singularities which make its investigation difficult, and are of no interest in the present analysis. First, owing to the noncompactness of C, there may be trajectories moving to infinity. For instance, a solid body may reach arbitrarily high angular velocities under the action of a constant moment. Such motion is impossible in practice owing to limited on-board energy resources. Second, the behavior of a six-dimensional dynamic system can be very complex, for example, hetero- and homoclinal structures may appear. To eliminate from the analysis these types of behavior we impose on the dynamic system a number of constraints.

Let the solid body be subjected to the action of moment $M(\omega,\lambda)$ which is smooth on $R^3\times P^3$ and satisfies the following conditions:

a) At fairly high angular velocities ω the quantity ωM is negative, i.e., when considering a dynamic system on a manifold with boundary $e^3\times P^3=\{\omega\in e^3,\lambda\in P^3\}$, where e^3 is a sphere of fairly large radius, the vector field at the boundary of that manifold $\partial(e^3\times P^3)=S^2\times P^3$ is assumed to be directed transversally inward the manifold.

b) The dynamic system is a generalized Morse - Smale system, i.e. it possesses the following properties:

b1) the set of nonstraying trajectories consists of a finite number of i-dimensional periodic surfaces $\beta_1,\beta_2,\ldots,\beta_n$ (an i-dimensional periodic surface is the invariant subset homeomorphic to the i-dimensional torus T^i, in particular the zero-dimensional periodic surface - a fixed point, or an equilibrium position-one-dimensional periodic surface - a closed trajectory, a two dimensional periodic surface - a two-dimensional torus which represent the union of trajectories);

b2) all β are hyperbolic;

b3) there are no closed cycles of trajectories among β_i, more exactly, a sequence of indices $i=i_1,i_2,\ldots,i_k=i$, such that $W^*(\beta_{i_j})\cap W(\beta_{i_{j+1}})\neq\varnothing$ for $1\leqslant j\leqslant k$, where $W(\beta_i)$, $W^*(\beta_i)$, is a stable and unstable manifold for β_i, does not exist.

The described dynamic system is defined on the manifold $e^3\times P^3$ with a boundary $S^2\times P^3$. To use the Morse—Smale inequalities it is necessary to get rid of the boundary. We join $e^3\times P^3$ with the second specimen of $e^3\times P^3$ along the border $S^2\times P^3$, and obtain the compact manifold D without boundary. Using standard procedures of differential topology /9/, it is possible, first to make D smooth and, second, calculate the homology groups D, and obtain Betti numbers modulo two

$$R_{0,1,2} = 1,\ R_3 = 2,\ R_{4,5,6} = 1 \tag{8}$$

It is now necessary to complete the definition of the vector field on the second specimen of $e^3 \times P^3$. We assume that a vector field with four critical points x_i such that $\dim W(x_i) = i,\ i = 0, 1, 2, 3$, is specified on it. For example, it is possible to assume that the vector field direction is opposite to that defined in the first part of this paper for the vector field of a dissipative dynamic system with the lowest possible number of fixed points. Smoothness of the gluid joint of vector fields of the first and second specimen of $e^3 \times P^3$ can be achieved by the usual procedure of small movement of the vector field in the neighborhood the gluid joint of $S^2 \times P^3$, using the transversality of vector fields at the boundary.

We use the generalized Morse – Smale inequalities for dynamic systems that are generally of the form

$$M_0 \geqslant R_0, \quad M_1 - M_0 \geqslant R_1 - R_0, \ldots, \quad \sum_{k=0}^{6} (-1)^k M_k = \sum_{k=0}^{6} (-1)^k R_k = \varkappa(D)$$

$$\left(M_s = \sum_{k=0}^{5} \sum_{i=0}^{k} \binom{k}{i} a_{s+i}^k,\ \binom{k}{i} = \frac{k!}{i!\,(k-i)!}\right)$$

where R_i is the i-th Betti number in modulo two of the manifold D, $\varkappa(D)$ is Euler's characteristic of D, a_j^0 is the number of fixed points β such that $\dim W(\beta) = j$, a_j^1 the number of closed trajectories of β such that $W(\beta) = j$, and a_j^i the number of i-dimensional periodic surfaces of β such that $\dim W(\beta) = j$ and $a_j^i = 0$ for $j < i$.

Taking into account that the dynamic system has already four fixed points x_i on D, we obtain

$$\begin{aligned}
&a_6^0 + a_6^1 + a_6^2 + a_6^3 + a_6^4 + a_6^5 \geqslant 1 \\
&a_5^0 - a_6^0 + a_5^1 + (a_5^2 + a_6^2) + (a_5^3 + 2a_6^3) + (a_5^4 + 3a_6^4) + \\
&\quad (a_5^5 + 4a_6^5) \geqslant 0 \\
&a_4^0 - a_5^0 + a_6^0 + a_4^1 + (a_4^2 + a_5^2) + (a_4^3 + 2a_5^3 + a_6^3) + \\
&\quad (a_4^4 + 3a_5^4 + 3a_6^4) + (4a_5^5 + 6a_6^5) \geqslant 1 \\
&a_3^0 - a_4^0 + a_5^0 - a_6^0 + a_3^1 + (a_3^2 + a_4^2) + (a_3^3 + 2a_4^3 + \\
&\quad a_5^3) + (3a_4^4 + 3a_5^4 + a_6^4) + \\
&\quad (6a_5^6 + 4a_6^5) \geqslant 0 \\
&a_2^0 - a_3^0 + a_4^0 - a_5^0 + a_6^0 + a_2^1 + (a_2^2 + a_3^2) + (2a_3^3 + \\
&\quad a_4^3) + (3a_4^4 + a_5^4) + (4a_5^5 + a_6^5) \geqslant 0 \\
&a_1^0 - a_2^0 + a_3^0 - a_4^0 + a_5^0 - a_6^0 + a_1^1 + a_2^2 + a_3^3 + \\
&\quad a_4^4 + a_5^5 \geqslant 0 \\
&a_0^0 - a_1^0 + a_2^0 - a_3^0 + a_4^0 - a_5^0 + a_6^0 = 0
\end{aligned} \tag{9}$$

The following statements follow from inequalities (9). If a solid body has only equilibrium positions, their number is not less than four; $a_k^0 \geqslant 1,\ k = 6, 5, 4, 3$; if there are only closed trajectories, their number is not less than two: $a_6^1 \geqslant 1,\ a_4^1 \geqslant 1$; when there are only two-dimensional tori, their number is not less than two: $a_6^2 \geqslant 1,\ a_4^2 + a_5^2 \geqslant 1$; and when there are only invarient i-dimensional tori: $(i = 3, 4, 5)$, at least one of them is stable: $a_6^i \geqslant 1$.

Thus in the case of a smooth moment the system of inequalities (9) define natural topological constraints on the dynamics of solid body rotations. Similar relations can, probably, be found for relay systems frequently encountered in applied mechanics.

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